MATHEMATICS

Section-1

1. If the mean and variance of the following data: 6, 10, 7, 13, a, 12, b, 12 are 9 and $\frac{37}{4}$ respectively, then $(a-b)^2$ is equal to:

A) 24 **B)** 12

C) 32 **D)** 16

Answer: D,
Explanation:

Mean $= \frac{6+10+7+13+a+12+b+12}{8} = 9$

$60+a+b=72$

$a+b=12....(1)$

Variance $= \frac{\sum x_i^2}{n} - \left(\frac{\sum x_i}{n}\right)^2 = \frac{37}{4}$

$\sum x_i^2 = 6^2+10^2+7^2+13^2+a^2+b^2+12^2+12^2$

$= a^2+b^2+642$

$\frac{a^2+b^2+642}{8} - (9)^2 = \frac{37}{4}$

$\frac{a^2+b^2}{8} + \frac{321}{4} - 81 = \frac{37}{4}$

$\frac{a^2+b^2}{8} = 81 + \frac{37}{4} - \frac{321}{4}$

$\frac{a^2+b^2}{8} = 81-71$

$\therefore a^2+b^2 = 80.....(2)$

From $(1)\ a^2+b^2+2ab=144$

$80+2ab=144 \therefore 2ab=64$

$(a-b)^2 = a^2+b^2-2ab = 80-64=16$

2. The value of $\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \frac{(2j-1)+8n}{(2j-1)+4n}$ is equal to :

A) $5+\log_e\left(\frac{3}{2}\right)$

B) $2-\log_e\left(\frac{2}{3}\right)$

C) $3+2\log_e\left(\frac{2}{3}\right)$

D) $1+2\log_e\left(\frac{3}{2}\right)$

Answer: D,

Explanation:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \frac{\left(\frac{2j}{n}-\frac{1}{n}+8\right)}{\left(\frac{2j}{n}-\frac{1}{n}+4\right)}$$

$$\int_0^1 \frac{2x+8}{2x+4}dx = \int_0^1 dx + \int_0^1 \frac{4}{2x+4}dx$$

$$= 1+4\frac{1}{2}\left(\ln|2x+4|\right)\Big|_0^1$$

$$= 1+2\ln\left(\frac{3}{2}\right)$$

3. Let $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + 2\hat{j} + 3\hat{k}$. Then the vector product
$\vec{\ } \ \vec{b}) \times ((\vec{a} \times ((\vec{a} - \vec{b}) \times \vec{b})) \times \vec{b})$ is equal to :

A) $5(34\hat{i} - 5\hat{j} + 3\hat{k})$

B) $7(34\hat{i} - 5\hat{j} + 3\hat{k})$

C) $7(30\hat{i} - 5\hat{j} + 7\hat{k})$

D) $5(30\hat{i} - 5\hat{j} + 7\hat{k})$

Answer: D,
Explanation:

$$\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$$

$$\vec{b} = -\hat{i} + 2\hat{j} + 3\hat{k}$$

$$\vec{a} + \vec{b} = 3\hat{j} + 5\hat{k}; \vec{a}.\vec{b} = -1 + 2 + 6 = 7$$

$$((\vec{a} \times ((\vec{a} - \vec{b}) \times \vec{b})) \times \vec{b})$$

$$((\vec{a} \times (\vec{a} \times \vec{b} - \vec{b} \times \vec{b})) \times \vec{b})$$

$$(\vec{a} \times (\vec{a} \times \vec{b} - 0)) \times \vec{b}$$

$$(\vec{a} \times (\vec{a} \times \vec{b})) \times \vec{b}$$

$$((\vec{a}.\vec{b})\vec{a} - (\vec{a}.\vec{a})\vec{b}) \times \vec{b}$$

$$(\vec{a}.\vec{b})\vec{a} \times \vec{b} - (\vec{a}.\vec{a})(\vec{b} \times \vec{b})$$

$$(\vec{a}.\vec{b})(\vec{a} \times \vec{b})$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 1 & 1 & 2 \\ -1 & 2 & 3 \end{vmatrix} = -\hat{i} - 5\hat{j} + 3\hat{k}$$

$$\therefore 7(-\hat{i} - 5\hat{j} + 3\hat{k})$$

$$(\vec{a} + \vec{b}) \times (7(-\hat{i} - 5\hat{j} + 3\hat{k}))$$

$$7(0\hat{i} + 3\hat{j} + 5\hat{k}) \times (-\hat{i} - 5\hat{j} + 3\hat{k})$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 3 & 5 \\ -1 & -5 & 3 \end{vmatrix}$$

$$\Rightarrow 34\hat{i} - (5)\hat{j} + (3\hat{k})$$

$$\Rightarrow 34\hat{i} - 5\hat{j} + 3\hat{k}$$

$$\therefore 7(34\hat{i} - 5\hat{j} + 3\hat{k})$$

4. The value of the definite integral $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{(1+e^{x\cos x})(\sin^4 x+\cos^4 x)}$

A) $-\frac{\pi}{2}$

B) $\frac{\pi}{2\sqrt{2}}$

C) $-\frac{\pi}{4}$

D) $\frac{\pi}{\sqrt{2}}$

Answer: B,

Explanation:

$$I=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{(1+e^{x\cos x})(\sin^4 x+\cos^4 x)} \ldots(1)$$

Using $\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$.

$$I=\int_{-\pi/4}^{\pi/4} \frac{dx}{(1+e^{-x\cos x})(\sin^4 x+\cos^4 x)}$$

Add (1) and (2)

$$2I=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{\sin^4 x+\cos^4 x}$$

$$2I=2\int_0^{\frac{\pi}{4}} \frac{(1+\tan^2 x)\sec^2 x}{\tan^4 x+1}dx$$

$$I=\int_0^{\frac{\pi}{4}} \frac{(1+\frac{1}{\tan^2 x})\sec^2 x}{(\tan x-\frac{1}{\tan x})^2+2}dx$$

$$\tan x-\frac{1}{\tan x}=t$$

$$\left(1+\frac{1}{\tan^2 x}\right)\sec^2 x dx=dt$$

$$I=\int_{-\infty}^{0}\frac{dt}{t^2+2}=\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)\right]_{-\infty}^{0}$$

$$I=0-\frac{1}{\sqrt{2}}\left(-\frac{\pi}{2}\right)=\frac{\pi}{2\sqrt{2}}$$

5. Let C be the set of all complex numbers. Let

$S1=\{z\in C\,|\,|z-3-2i|^2=8\}$,

$S_2=\{z\in C\,|\,\mathrm{Re}(z)>5\}$ and

$S_3=\{z\in C\,|\,|z-z|\geq 8\}$.

Then the number of elements in $S_1\cap S_2\cap S_3$ is equal to

A) 1

B) 0

C) 2

D) Infinite

Answer: A,

Explanation:

$S_1:|z-3-2i|^2=8$

$|z-3-2i|=2\sqrt{2}$

$(x-3)^2+(y-2)^2=(2\sqrt{2})^2$

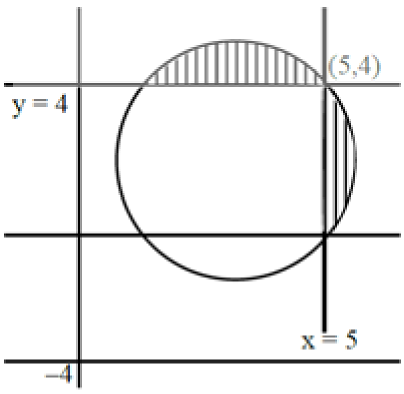
$S_2:x\geq 5$

$S_3:|z-\bar{z}|\geq 8$

$|2iy|\geq 8$

$2|y|\geq 8$

$\therefore y\geq 4, y\leq -4$



$n(S_1\cap S_2\cap S_3)=1$

6. If the area of the bounded region $R=\left\{(x,y): \max\{0,\log_e x\} \le y \le 2^x, \frac{1}{2} \le x \le 2\right\}$ is, $\alpha(\log_e 2)^{-1}+\beta(\log_e 2)+\gamma$, then the value of $(\alpha+\beta-2\gamma)^2$ is equal to :

A) 8

B) 2

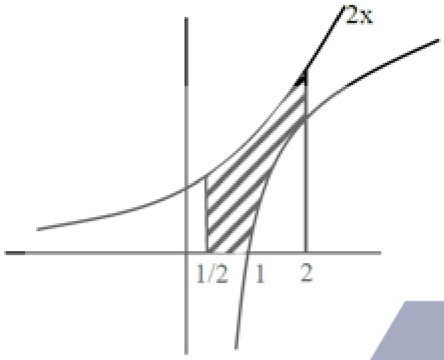
C) 4

D) 1

Answer: B,

Explanation:

$R=\left\{(x,y): \max(0,\log_e x) \le y \le 2^x, \frac{1}{2} \le x \le 2\right\}$



$$\int_{\frac{1}{2}}^{2} 2^x dx - \int_{1}^{2} \ln x\, dx$$

$$\Rightarrow \left[\frac{2^x}{\ln 2}\right]_{1/2}^{2} - [x\ln x - x]_1^2$$

$$\Rightarrow \frac{(2^2)-2^{1/2}}{\log_e 2} - (2\ln 2 - 1)$$

$$\Rightarrow \frac{(2^2-\sqrt{2})}{\log_e 2} - 2\ln 2 + 1$$

$$\therefore \alpha = 2^2-\sqrt{2}, \beta = -2, \gamma = 1$$

$$\Rightarrow (\alpha+\beta+2\gamma)^2$$

$$\Rightarrow (2^2-\sqrt{2}-2-2)^2$$

$$\Rightarrow (\sqrt{2})^2 = 2$$

7. A ray of light through (2,1) is reflected at a point P on the y-axis and then passes through the point (5, 3). If this reflected ray is the directrix of an ellipse with eccentricity $\frac{1}{3}$ and the distance of the nearer focus from this directrix is $\frac{8}{\sqrt{53}}$, then the equation of the other directrix can be:

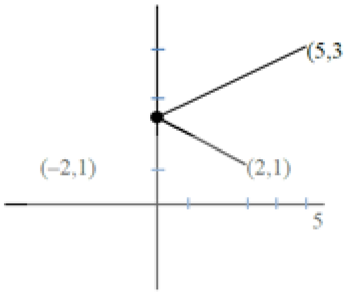
A) $11x + 7y + 8 = 0$ or $11x + 7y - 15 = 0$

B) $11x - 7y - 8 = 0$ or $11x + 7y + 15 = 0$

C) $2x - 7y + 29 = 0$ or $2x - 7y - 7 = 0$

D) $2x - 7y - 39 = 0$ or $2x - 7y - 7 = 0$

Answer: C,
Explanation:



Equation of reflected Ray

$$y-1=\frac{2}{7}(x+2)$$

$$7y-7=2x+4$$

$$2x-7y+11=0$$

Let the equation of other directrix is

$$2x-7y+\lambda$$

Distance of directrix from Focus

$$\frac{a}{e}-ae=\frac{8}{\sqrt{53}}$$

$$3a-\frac{a}{3}=\frac{8}{\sqrt{53}} \text{ or } a=\frac{3}{\sqrt{53}}$$

Distance from other focus $\frac{a}{e}+ae$

$$3a+\frac{a}{3}=\frac{10}{a}=\frac{10}{3}\times\frac{3}{\sqrt{53}}=\frac{10}{\sqrt{53}}$$

Distance between two directrix $=\frac{2a}{e}$

$$=2\times3\times\frac{3}{\sqrt{53}}=\frac{18}{\sqrt{53}}$$

$$\left|\frac{\lambda-11}{\sqrt{53}}\right|=\frac{18}{\sqrt{53}}$$

$\lambda - 11 = 18 \; or - 18$

$\lambda = 29 \; or \; -7$

$2x - 7y - 7 = 0 \; or \; 2x - 7y + 29 = 0$

8. If the coefficients of x^7 in $\left(x^2 + \frac{1}{bc}\right)^{11}$ and x^{-7} in $\left(x - \frac{1}{bx^2}\right)^{11}, b \neq 0$, are equal, then the value of b is equal to:

A) 2

B) -1

C) 1

D) -2

Answer: C,
Explanation:

Coefficient of x^7 **in** $\left(x^2 + \frac{1}{bx}\right)^{11}$

$_{11}C_r (x^2)^{11-r} . \left(\frac{1}{bx}\right)^2$

$_{11}C_r x^{22-3r} . \frac{1}{b^r}$

$22 - 3r = 7$

$r = 5$

$\therefore {}^{11}C_5 . \frac{1}{b^5} . x^7$

Coefficient of x^{-7} **in** $\left(x^2 - \frac{b}{bx^2}\right)^{11}$

${}^{11}C_r x^{11-3r} . \frac{(-1)^r}{b^r}$

$11 - 3r = -7$

$\therefore r = 6$

${}^{11}C_6 . \frac{1}{b^6} x^{-7}$

${}^{11}C_5 \frac{1}{b^5} = {}^{11}C_6 . \frac{1}{b^6}$

Since $b \neq 0 \therefore b = 1$

9. The compound statement $(P \vee Q) \wedge (\sim P) \Rightarrow Q$ is equivalent to:

A) $P \vee Q$

B) $P \wedge \sim Q$

C) $\sim (P \Rightarrow Q)$

D) $\sim (P \Rightarrow Q) \Leftrightarrow P \wedge \sim Q$

Answer: D,
Explanation:
Using Truth Table

P	Q	$P \vee Q$	$\sim P$	$(P \vee Q) \wedge P$	$(P \vee Q) \wedge \sim P \rightarrow Q$
T	T	T	F	F	T
T	F	T	F	F	T
F	T	T	T	T	T
F	F	F	T	F	T

P	Q	$\sim Q$	$P \wedge \sim Q$	$P \rightarrow Q$	$\sim (P \rightarrow Q)$
T	T	F	F	T	F
T	F	T	T	F	T
F	T	F	F	T	F
F	F	T	F	T	F

$\sim (P \rightarrow Q)$	$P \wedge \sim Q$	$\sim (P \rightarrow Q) \Leftrightarrow P \wedge \sim Q$
F	F	T
T	T	T
F	F	T
F	F	T

10. If $\sin\theta+\cos\theta=\frac{1}{2}$, then $16(\sin(2\theta)+\cos 4\theta+\sin(6\theta))$ is equal to:

A) 23 **B)** -27

C) -23 **D)** 27

Answer: C,
Explanation:

$$\sin\theta+\cos\theta=\frac{1}{2}$$

$$\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta=\frac{1}{4}$$

$$\sin 2\theta=-\frac{3}{4}$$

Now:

$$\cos 4\theta=1-2\sin^2 2\theta$$

$$=1-2\left(-\frac{3}{4}\right)^2$$

$$=1-2\times\frac{9}{16}=-\frac{1}{8}$$

$$\sin 6\theta=3\sin 2\theta-4\sin^3 2\theta$$

$$=(3-4\sin^2 2\theta).\sin 2\theta$$

$$=\left[3-4\left(\frac{9}{16}\right)\right].\left(-\frac{3}{4}\right)$$

$$\Rightarrow\left[\frac{3}{4}\right]\times\left(-\frac{3}{4}\right)=-\frac{9}{16}$$

$$16[\sin 2\theta+\cos 4\theta+\sin 6\theta]$$

$$16\left(-\frac{3}{4}-\frac{1}{8}-\frac{9}{16}\right)=-23$$

Section-2

11. Let $A=\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$. If $A^{-1}=\alpha I+\beta A, \alpha,\beta \in R, I$ is a 2×2 identity matrix, then $4(\alpha-\beta)$ is equal to :

A) 5

B) $\frac{8}{3}$

C) 2

D) 4

Answer: D,

Explanation:

$$A=\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}, |A|=6$$

$$A^{-1}=\frac{adjA}{|A|}=\frac{1}{6}\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}=\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix}$$

$$\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix}=\begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}+\begin{bmatrix} \beta & 2\beta \\ -\beta & 4\beta \end{bmatrix}$$

$$\left.\begin{aligned} \alpha+\beta &= \frac{2}{3} \\ \beta &= -\frac{1}{6} \end{aligned}\right\} \Rightarrow \alpha=\frac{2}{3}+\frac{1}{6}=\frac{5}{6}$$

$$4(\alpha-\beta)=4(1)=4$$

12. Let $f:\left(-\frac{\pi}{4},\frac{\pi}{4}\right)\to R$ be defined as

$$f(x)=\begin{cases}(1+|\sin x|)^{\frac{3a}{|\sin x|}} & , -\frac{\pi}{4}<x<0\\ b & , x=0\\ e^{\cot 4x/\cot 2x} & , 0<x<\frac{\pi}{4}\end{cases}$$

If f is continuous at x = 0, then the value of $6a+b^2$ is equal to:

A) 1 – e **B)** e –1

C) 1 + e **D)** e

Answer: C,
Explanation:

$$\lim_{x\to 0} f(x)=b$$

$$\lim_{x\to 0^+} xe^{\frac{\cot 4x}{\cot 2x}}=e^{\frac{1}{2}}=b$$

$$\lim_{x\to 0^-}(1+|\sin x|)^{\frac{3a}{|\sin x|}}=e^{3a}=e^{\frac{1}{2}}$$

$$\lim_{x\to 0^-}(1+|\sin x|)^{\frac{3a}{|\sin x|}}=e^{3a}=e^{\frac{1}{2}}$$

$$a=\frac{1}{6}\Rightarrow 6a=1$$

$$(6a+b^2)=(1+e)$$

13. Let y = y(x) be solution of the differential equation $\log_e\left(\frac{dy}{dx}\right)=3x+4y$, with y(0) = 0.

If $y\left(-\frac{2}{3}\log_e 2\right)=\alpha\log_e 2$, then the value of α is equal to:

A) $-\frac{1}{4}$

B) $\frac{1}{4}$

C) 2

D) $-\frac{1}{2}$

Answer: A,

Explanation:

$$\frac{dy}{dx}=e^{3x}.e^{4y}\Rightarrow\int e^{-4y}dy=\int e^{3x}dx$$

$$\frac{e^{-4y}}{-4}=\frac{e^{3x}}{3}+C\Rightarrow-\frac{1}{4}-\frac{1}{3}=C\Rightarrow C=-\frac{7}{12}$$

$$\frac{e^{-4y}}{-4}=\frac{e^{3x}}{3}-\frac{7}{12}\Rightarrow e^{-4y}=\frac{4e^{3x}-7}{-3}$$

$$e^{4y}=\frac{3}{7-4e^{3x}}\Rightarrow 4y=\ln\left(\frac{3}{7-4e^{3x}}\right)$$

$$4y=\ln\left(\frac{3}{6}\right)\textit{when } x=-\frac{2}{3}\ln 2$$

$$y=\frac{1}{4}\ln\left(\frac{1}{2}\right)=-\frac{1}{4}\ln 2$$

14. Let the plane passing through the point (–1, 0, –2) and perpendicular to each of the planes 2x + y – z = 2 and x – y – z = 3 be ax + by + cz + 8 = 0. Then the value of a + b + c is equal to:

A) 3 **B)** 8

C) 5 **D)** 4

Answer: D,
Explanation:

Normal of req . plane $(2\hat{i}+\hat{j}-\hat{k})\times(\hat{i}-\hat{j}-\hat{k})$

$=-2\hat{i}+\hat{j}-3\hat{k}$

Equation of plane

$-2(x+1)+1(y-0)-3(z+2)=0$

$-2x+y-3z-8=0$

$2x-y+3z+8=0$

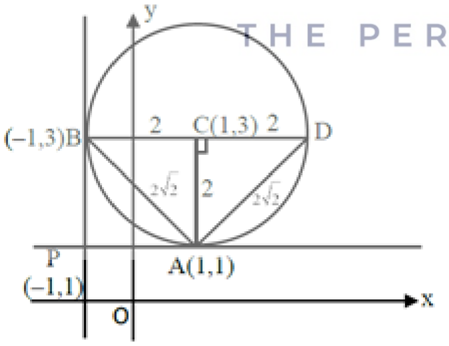
$a+b+c=4$

15. Two tangents are drawn from the point P(–1, 1) to the circle $x^2+y^2-2x-6y+6=0$. If these tangents touch the circle at points A and B, and if D is a point on the circle such that length of the segments AB and AD are equal, then the area of the triangle ABD is equal to:

A) 2 **B)** $(3\sqrt{2}+2)$

C) 4 **D)** $(3\sqrt{2}-1)$

Answer: C,
Explanation:



$\Delta ABD=\frac{1}{2}\times2\times4$

$=4$

16. Let $f: R \rightarrow R$ be a function such that f(2) = 4 and f'(2) =1. Then, the value of $\lim_{x \to 2} \frac{x^2 f(2) - 4f(x)}{x-2}$ is equal to

A) 4
B) 8
C) 16
D) 12

Answer: D,
Explanation:
Apply L'Hopital Rule

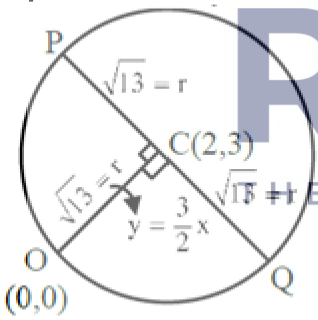
$$\lim_{x \to 2} \left(\frac{2xf(2) - 4f'(x)}{1} \right)$$

$$= \frac{4(4)-4}{1} = 12$$

17. Let P and Q be two distinct points on a circle which has center at C(2, 3) and which passes through origin O. If OC is perpendicular to both the line segments CP and CQ, then the set {P, Q} is equal to

A) $\{(4,0),(0,6)\}$
B) $\{(2+2\sqrt{2},3-\sqrt{5}),(2-2\sqrt{2},3+\sqrt{5})\}$
C) $\{(2+2\sqrt{2},3+\sqrt{5}),(2-2\sqrt{2},3-\sqrt{5})\}$
D) $\{(-1, 5), (5,1)\}$

Answer: D,
Explanation:



$$\tan\theta = -\frac{2}{3}$$

Using symmetric from of line

$$P,Q:(2 \pm \sqrt{13}\cos\theta, 3 \pm \sqrt{13}\sin\theta)$$

$$\left(2 \pm \sqrt{13}.\left(-\frac{3}{\sqrt{13}}\right), 3 \pm \sqrt{13}\left(\frac{2}{\sqrt{13}}\right)\right)$$

$$(-1,5)\&(5,1)$$

18. Let α, β be two roots of the equation $x^2 + (20)^{1/4}x + (5)^{1/2} = 0$. Then $\alpha^8 + \beta^8$ is equal to

A) 10 **B)** 100

C) 50 **D)** 160

Answer: C,
Explanation:

$(x^2 + \sqrt{5})^2 = \sqrt{20}x^2$

$x^4 = -5 \Rightarrow x^8 = 25$

$\alpha^8 + \beta^8 = 50$

19. The probability that a randomly selected 2-digit number belongs to the set $\{n \in N : (2^n - 2) \text{ is a multiple of } 3\}$ is equal to

A) $\frac{1}{6}$ **B)** $\frac{2}{3}$

C) $\frac{1}{2}$ **D)** $\frac{1}{3}$

Answer: C,
Explanation:

Total number of cases $= {}^{90}C_1 = 90$

Now, $2^n - 2 = (3-1)^n - 2$

${}^nC_0 3^n - {}^nC_1.3^{n-1} + ... + (-1)^{n-1}.{}^nC_{n-1}3 + (-1)^n.{}^nC_n - 2$

$3(3^{n-1} - n3^{n-2} + ... + (-1)^{n-1}.n) + (-1)^n - 2$

$(2^n - 2)$ **is multiply of 3 only when n is odd**

Req. Probability $= \frac{45}{90} = \frac{1}{2}$

20. Let

$A=\{(x,y)\in R\times R \mid 2x^2+2y^2-2x-2y=1\}$

$B=\{(x,y)\in R\times R \mid 4x^2+4y^2-16y+7=0\}$

$C=\{(x,y)\in R\times R \mid x^2+y^2-4x-2y+5\le r^2\}$.

Then the minimum value of $|r|$ such that $A\cup B\subseteq C$ is equal to

A) $\frac{3+\sqrt{10}}{2}$

B) $\frac{2+\sqrt{10}}{2}$

C) $\frac{3+2\sqrt{5}}{2}$

D) $1+\sqrt{5}$

Answer: C,

Explanation:

$$S_1: x^2+y^2-x-y-\frac{1}{2}=0; C_1\left(\frac{1}{2},\frac{1}{2}\right)$$

$$r_1=\sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{2}}=1$$

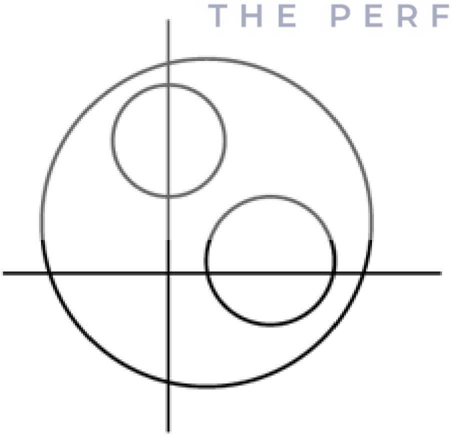
$$S_2: x^2+y^2-4y+\frac{7}{4}=0; C_2:(0,2)$$

$$r_2=\sqrt{4-\frac{7}{4}}=\frac{3}{2}$$

$$S_3: x^2+y^2-4x-2y+5-r^2=0$$

$$C_3:(2,1)$$

$$r_3=\sqrt{4+1-5+r^2}=|r|$$



$$C_1C_3=\sqrt{\frac{5}{2}}$$

$$\sqrt{\frac{5}{2}}\le|r-1| \Rightarrow \left.\begin{array}{l} r\le 1+\sqrt{\frac{5}{2}} \\ r\ge\frac{3}{2}+\sqrt{5}\end{array}\right\}$$

$$C_2C_3 = \sqrt{5} \le \left|r - \frac{3}{2}\right|$$

$$\left.\begin{array}{r} r - \dfrac{3}{2} \ge \sqrt{5} \\ r - \dfrac{3}{2} \le -\sqrt{5} \end{array}\right\}$$

21. **Let** $\vec{a} = \hat{i} + \hat{j} + \hat{k}, \vec{b}$ **and** $\vec{c} = \hat{j} - \hat{k}$ **be three vectors such that** $\vec{a} \times \vec{b} = \vec{c}$ **and** $\vec{a} \cdot \vec{b} = 1$. **If the length of projection vector of the vector** $\vec{b}$ **on the vector** $\vec{a} \times \vec{c}$ **is** ℓ**, then the value of** $3l^2$ **is equal to _______.**

Answer: ____________________

Answer: 2

Explanation:

$\vec{a} \times \vec{b} = c$

Take Dot with $\vec{c}$

$(\vec{a} \times \vec{b}).\vec{c} = |\vec{c}|^2 = 2$

Projection of $\vec{b}$ **or** $\vec{a} \times \vec{c} = l$

$$\frac{|\vec{b}.(\vec{a} \times \vec{c})|}{|\vec{a} \times \vec{c}|} = l$$

$$\therefore l = \frac{2}{\sqrt{6}} \Rightarrow l^2 = \frac{4}{6}$$

$3l^2 = 2$

22. If $\log_3 2, \log_3(2^x - 5), \log_3\left(2^x - \frac{7}{2}\right)$ are in an arithmetic progression, then the value of x is equal to ________.

Answer: ___________________

Answer: 3

Explanation:

$$2\log_3(2^x - 5) = \log_3 2 + \log_3\left(2^x - \frac{7}{2}\right)$$

Let $2^x = t$

$$\log_3(t-5)^2 = \log_3 2\left(t - \frac{7}{2}\right)$$

$$(t-5)^2 = 2t - 7$$

$$t^2 - 12t + 32 = 0$$

$$(t-4)(t-8) = 0$$

$$\Rightarrow 2^x = 4 \text{ or } 2^x = 8$$

$X = 2$ (Rejected)

Or $x = 3$

23. Let the domain of the function $f(x) = \log_4(\log_5(\log_3(18x - x^2 - 77)))$ be (a, b). Then the value of the integral $\int_a^b \frac{\sin^3 x}{(\sin^3 x + \sin(a+b-x))} dx$ is equal to ________.

Answer: ____________________

Answer: 1

Explanation:

For domain

$\log_5(\log_3(18x - x^2 - 77)) > 0$

$\log_3(18x - x^2 - 77) > 1$

$18x - x^2 - 77 > 3$

$x^2 - 18x + 80 < 0$

$x \in (8, 10)$

$\Rightarrow a = 8 \text{ and } b = 10$

$$I = \int_a^b \frac{\sin^3 x}{\sin^3 x + \sin^3(a+b-x)} dx$$

$$I = \int_a^b \frac{\sin^3 x(a+b-x)}{\sin^3 x + \sin^3(a+b-x)}$$

$$2I = (b-a) \Rightarrow I = \frac{b-a}{2} (\because a = 8 \text{ and } b = 10)$$

$$I = \frac{10-8}{2} = 1$$

24. Let $f(x) = \begin{vmatrix} \sin^2 x & -2+\cos^2 x & \cos 2x \\ 2+\sin^2 x & \cos^2 x & \cos 2x \\ \sin^2 x & \cos^2 x & 1+\cos 2x \end{vmatrix}, x \in [0, \pi]$ Then the maximum value of f(x) is equal to ________.

Answer: ____________________

Answer: 6

Explanation:

$$\begin{vmatrix} -2 & -2 & 0 \\ 2 & 0 & -1 \\ \sin^2 x & \cos^2 1 & 1+\cos 2x \end{vmatrix} \begin{pmatrix} R_1 \to R_1 - R_2 \\ \& \ R_2 \to R_2 - R_3 \end{pmatrix}$$

$-2(\cos^2 x) + 2(2 + 2\cos 2x + \sin^2 x)$

$4 + 4\cos 2x - 2(\cos^2 x - \sin^2 x)$

$f(x) = 4 + \underbrace{2\cos 2x}_{\max = 1}$

$f(x)_{max} = 4 + 2 = 6$

25. Let $F:[3,5] \to R$ be a twice differentiable function on (3, 5) such that $F(x) = e^{-x}\int_3^x (3t^2 + 2t + 4F'(t))dt$. If $F'(4) = \frac{\alpha e^{\beta} - 224}{(e^{\beta} - 4)^2}$, then α+β is equal to _________.

Answer: ____________________

Answer: 16

Explanation:

$F(3) = 0$

$$e^x F(x) = \int_3^x (3t^2 + 2t + 4F'(t))\,dt$$

$$e^x F(x) + e^x F'(x) = 3x^2 + 2x + 4F'(x)$$

$$(e^x - 4)\frac{dy}{dx} + e^x y = (3x^2 + 2x)$$

$$\frac{dy}{dx} + \frac{e^x}{(e^x - 4)} y = \frac{(3x^2 + 2x)}{(e^x - 4)}$$

$$y e^{\int \frac{e^x}{(e^x - 4)}dx} = \int \frac{(3x^2 + 2x)}{(e^x - 4)} e^{\int \frac{e^x}{e^x - 4}dx} dx$$

$$y.(e^x - 4) = \int (3x^2 + 2x)dx + c$$

$$y(e^x - 4) = x^3 + x^2 + c$$

$$\text{Put } x = 3 \Rightarrow c = -36$$

$$F(x) = \frac{(x^3 + x^2 - 36)}{(e^x - 4)}$$

$$F'(x) = \frac{(3x^2 + 2x)(e^x - 4) - (x^3 + x^2 - 36)e^x}{(e^x - 4)^2}$$

$$F'(4) = \frac{56(e^4 - 4) - 4ne^n}{(e^4 - 4)^2}$$

$$= \frac{12e^4 - 22y}{(e^y - 4)^2} \Rightarrow \alpha = 12$$

$$\beta = 4$$

$$\alpha + \beta = 16$$

26. Let a plane P pass through the point (3, 7, –7) and contain the line, $\frac{x-2}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$. If distance of the plane P from the origin is d, then d^2 is equal to _________.

Answer: ____________________

Answer: 3

Explanation:

$\overrightarrow{BA}=(\hat{i}+4\hat{j}-5\hat{k})$

$$\overrightarrow{BA}\times\vec{l}=\vec{n}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -3 & 2 & 1\\ 1 & 4 & -5\end{vmatrix}$$

$a\hat{i}+b\hat{j}+c\hat{k}=-14\hat{i}-\hat{j}(m)+\hat{k}(-14)$

$a=1, b=1, c=1$

Plane is $(x-2)+(y-3)+(z+z)=0$

$x+y+z-3=0$

$d=\sqrt{3}\Rightarrow d^2=3$

27. Let S = {1, 2, 3, 4, 5, 6, 7}. Then the number of possible functions $f : S \to S$ such that f(m·n) = f(m)·f(n) for every $m, n \in S$ and $m.n \in S$ is equal to______.

Answer: ____________________

Answer: 490

Explanation:

$F(mn) = f(m).f(n)$

Put m = 1 f(n) = f(1).f(n) ⇒ f(1) = 1

$Put\ m = n = 2$

$$f(4) = f(2).f(2)\begin{cases} f(2) = 1 \Rightarrow f(4) = 1 \\ or \\ f(2) = 2 \Rightarrow f(4) = 4 \end{cases}$$

$put\ m = 2, n = 3$

$$f(6) = f(2).f(3)\begin{cases} when\ f(2) = 1 \\ f(3) = 1\ to\ 7 \\ f(2) = 2 \\ f(3) = 1\ or\ 2\ or\ 3 \end{cases}$$

f (5), f (7) can take any value

Total $|x| \times 7 \times 1 \times 7 \times 1 \times 7$

$|x| \times 3 \times 1 \times 7 \times 1 \times 7$

$= 490$

28. If $y = y(x), y \in [0, \frac{\pi}{2})$ is the solution of the differential equation $\sec y \frac{dy}{dx} - \sin(x+y) - \sin(x-y) = 0$, with $y(0) = 0$, then $5y'\left(\frac{\pi}{2}\right)$ is equal to __________.

Answer: ____________________

Answer: 2

Explanation:

$$\sec y \frac{dy}{dx} = 2\sin x \cos y$$

$$\sec^2 y dy = 2\sin x dx$$

$$\tan y = -2\cos x + c$$

$$c = 2$$

$$\tan y = -2\cos x + 2 \Rightarrow at\ x = \frac{\pi}{2}$$

$$\tan y = 2$$

$$\sec^2 y \frac{dy}{dx} = 2\sin x$$

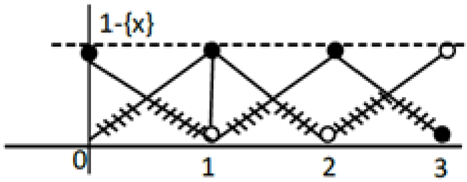
$$5\frac{dy}{dx} = 2$$

29. Let $f:[0,3]\to R$ be defined by $f(x)=\min\{x-[x],1+[x]-x\}$ where [x] is the greatest integer less than or equal to x. Let P denote the set containing all $x\in[0,3]$ where f is discontinuous, and Q denote the set containing all $x\in(0,3)$ where f is not differentiable. Then the sum of number of elements in P and Q is equal to ________.

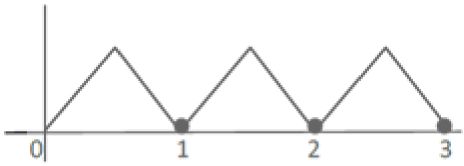
Answer: ____________________

Answer: 5

Explanation:



$1-\{x\}=1-x;0\le x<1$



Non differentiable at

$x=\frac{1}{2},1,\frac{3}{2},2,\frac{5}{2}$

30. For real numbers α and β, consider the following system of linear equations :

$x+y-z=2,\ x+2y+\alpha z=1, 2x-y+z=\beta$.

If the system has infinite solutions, then $\alpha+\beta$ is equal to ________

Answer: ____________________

Answer: 5

Explanation:

For infinite solutions

$$\Delta=\Delta_1=\Delta_2=\Delta_3=0$$

$$\Delta=\begin{vmatrix}1 & 1 & -1\\ 1 & 2 & \alpha\\ 2 & -1 & 1\end{vmatrix}=0$$

$$\Delta=\begin{vmatrix}3 & 0 & 0\\ 1 & 2 & \alpha\\ 2 & -1 & 1\end{vmatrix}=0$$

$$\Delta=3(2+\alpha)=0$$

$$\Rightarrow \alpha=-2$$

$$\Delta_2=\begin{vmatrix}1 & 2 & -1\\ 1 & 1 & -2\\ 2 & \beta & 1\end{vmatrix}=0$$

$$1(1+2\beta)-2(1+4)-(\beta-2)=0$$

$$\beta-7=0$$

$$\beta=7$$

$$\therefore \alpha+\beta=5$$