22-07-2021 JEE MAINS (SHIFT – 2) PAPER – 1 MATHEMATICS MEMORY BASED

1. If $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and B be a 3×3 matrix whose entries are $\{1, 2, 3, 4, 5\}$, then the number of possible matrices B such that $AB = BA$ are:

   A. 240

   B. 320

   C. 120

   D. 100

Sol: $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Let $B = \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$

Now $AB = BA$

$\Rightarrow \begin{bmatrix} p & q & r \\ a & b & c \\ x & y & z \end{bmatrix} - \begin{bmatrix} b & a & c \\ q & p & r \\ y & x & z \end{bmatrix}$

$\Rightarrow p = b, a = q, r = c, x = y \,\&\, z = z$

Hence numbers of such matrices are $5! = 120$

2. The number of all possible numbers less than 10000 that can be formed using the digits 0, 2, 4, 6, 8 (without repetition) are:

   Ans: 165

   Sol: 1 digit number = 5

   2 digit number = 4.4 = 16

   3 digit number = 4.4.3 = 48

   4 digit number = 4.4.3.2 = 96

   Total = 5 +16 +48 +96 = 165

3. If the circle $36x^2 + 36y^2 - 108x + 120y + c = 0$ neither touches nor cuts the co-ordinate axis and point of intersection of lines $x - 2y = 4$ and $2x - y = 5$ also lies inside the circle, then the value of c lies in:

   A. 81 < c < 156

   B. 100 < c < 156

   C. 81 < c < 150

   D. 100 < c < 150

Sol: Intersection points of $2x - y = 5$ and $x - 2y = 4$ is $(2, -1)$

So, $(2, -1)$ lies inside the circle $\Rightarrow S_1 < 0$

$36(2)^2 + 36(-1)^2 - 108(2) + 120(-1) + c < 0$

$c < 156 \quad \ldots.(i)$

Circle $36x^2 + 36y^2 - 108x + 120y + c = 0$ neither touches nor cuts the co-ordinate axis

So, $g^2 - c < 0 \Rightarrow \left(\frac{3}{2}\right)^2 - \frac{c}{36} < 0 \Rightarrow c > 81 \ldots (ii)$

And $f^2 - c < 0 \Rightarrow \left(\frac{5}{3}\right)^2 - \frac{c}{36} < 0 \Rightarrow c > 100 \ldots (iii)$

4. If line $2x + y = k, (k < 0)$ is a tangent to both the curves $x^2 - y^2 = 3$ and $y^2 = \alpha x$, then the value of α is

Ans: 24.00

Sol: Given slope of line $(m) = -2$

Slope form of tangent to the curve $x^2 - y^2 = 3$ is $y = mx \pm \sqrt{a^2m^2 - b^2}$

$$\Rightarrow y = -2x \pm 3$$

On comparing, with the equation $2x + y = k, (k < 0) \Rightarrow k = -3$

Now, slope form of tangent to the parabola $y^2 = \alpha x$ is $y^2 = mx + \frac{\alpha}{4m}$

But $m = -2$ so, $y = -2x + \frac{\alpha}{4(-2)}$

$\Rightarrow -3 = \frac{\alpha}{4(-2)}$

$\alpha = 24$

5. The number of all possible values of $n \in \{1, 2, 3, \ldots, 100\}$ which satisfy the condition $11^n > 10^n + 9^n$ is:

Ans: 96

Sol: let $11^n > 10^n + 9^n$, $n \in \{1, 2, 3, \ldots, 100\}$

$\Rightarrow 11^n - 9^n > 10^n$

$\Rightarrow (10 + 1)^n - (10 - 1)^n > 10^n$

$\Rightarrow 2\left[{}^nC_1 10^{n-1} + {}^nC_3 10^{n-3} + {}^nC_5 10^{n-5} + \cdots\right] > 10^n$

$\Rightarrow \frac{1}{5}\left[{}^nC_1 10^{n} + {}^nC_3 10^{n-2} + {}^nC_5 10^{n-4} + \cdots\right] > 10^n$

$\Rightarrow \frac{1}{5}\left[{}^nC_1 + {}^nC_3 10^{-2} + {}^nC_5 10^{-4} + \cdots\right] > 1$

Clearly the above inequality is true for $n > 5$

For $n = 4$ we have $\frac{1}{5}\left[4 + \frac{4}{10^2}\right] = \frac{4}{5}\left(\frac{101}{100}\right) < 1$ (rejected)

Hence Number of such $n \in \{1, 2, 3, \ldots, 100\}$ is equal to 96.

6. If an AP, $S_{10} = 530$ & $S_5 = 140$ {Where S_n denotes the sum of first n terms of an AP}, then the value of $S_{20} - S_6$ is

Ans: 2

Sol: $S_{10} = 530$

$\frac{10}{2}[2a + 9d] = 530$

$2a + 9d = 106 \ \ldots(1)$

$S_5 = 140$

$\frac{5}{2}[2a + 4d] = 140$

$2a + 9d = 56 \ \ldots(2)$

$5d = 50, d = 10, a = 8$

Now, $S_{20} - S_6$

$10\ [2a + 19d] - 3[2a + 5d]$

$14a + 175d$

$14 \times 8 + (175)10 = 1862$

7. The value of r for which the term independent of x in the expansion of $\left(2x^r + \frac{1}{x^2}\right)^{10}$ is 180 is:

A. 7

B. 8

C. 9

D. 10

Sol:$T_{k+1} = 10_{C_k}(2x^r)^{10-k}(x)^{-2x}$

$\Rightarrow\ 10_{C_k}2^{(10-k)}.x^{10r-rk-2k}$

Now, $10r - rk - 2k = 0\ \Rightarrow r = \frac{2k}{10-k}$

And $10_{C_k}2^{(10-k)} = 180\ \Rightarrow k = 8$

8. If $z^2 + 3\bar{z} = 0$ has n solutions, then the value of $\sum_{k=0}^{\infty}\frac{1}{n^k}$ is:

A. $\frac{3}{4}$

B. $\frac{4}{3}$

C. $\frac{5}{5}$

D. $\frac{1}{2}$

Sol: Let $z = x + iy$

$(x + iy)^2 + 3(x - iy) = 0$

$x^2 - y^2 + 3x = 0\ \&\ 2xy - 3y = 0$

Case 1: $y = 0$

$x^2 - y^2 + 3x = 0$

$\Rightarrow x = 0 \text{ or } x = -3$

Case 2: $x = \frac{3}{2}$

$x^2 - y^2 + 3x = 0$

$y = \frac{3\sqrt{3}}{2}$ or $y = \frac{-3\sqrt{3}}{2}$

Solutions are $z = \frac{3}{2} + i\frac{3\sqrt{3}}{2}$ and $z = \frac{3}{2} - i\frac{3\sqrt{3}}{2}$

Total number of solutions = n = 4

So, $\sum_{k=0}^{\infty} \frac{1}{4^k} = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$

9. The number of solution of the equation $\sin^7 x + \cos^7 x = 1$ in the interval $[0, 4\pi]$ is:

A. 7

B. 9

C. 14

D. 5

Sol: $\sin^2 x + \cos^2 x = 1$, $\sin^2 x \le 1$ and $\cos^2 x \le 1$

$\sin^7 x \le \sin^2 x$, $\cos^7 x \le \cos^2 x$

So, $\sin^7 x + \cos^7 x \le 1$

$\sin^7 x + \cos^7 x = 1$

When $\sin^7 x = \sin^2 x$, $\cos^7 x = \cos^2 x$

Case – 1: $\sin x = 0$, $\cos x = 1 \Rightarrow x = 0, 2\pi, 4\pi$

Case – 2: $\sin x = 1$, $\cos x = 0 \Rightarrow x = \frac{\pi}{2}, \frac{5\pi}{2}$

Total Number of solution = 5

10. $f(x) = \begin{cases} -4\frac{x^3}{3} + 2x^2, x > 0 \\ 3xe^x, \ x \le 0 \end{cases}$ is minimum at

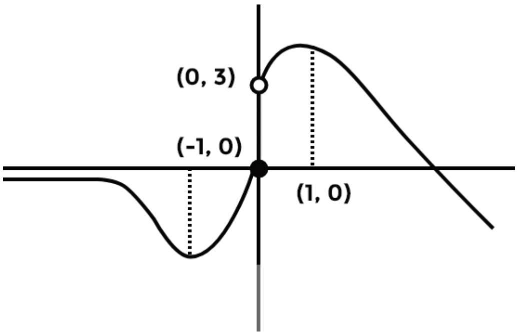
A. $\left(-1, \frac{3}{2}\right)$

B. $(0, 1)$

C. $\left(\frac{-3}{2}, 1\right)$

D. $\left(\frac{1}{2}, 2\right)$

Sol: $f'(x) = \begin{cases} -4x^2 + 4x, x > 0 \\ 3(xe^x + e^x),\ x \le 0 \end{cases}$



11. The sum of all natural numbers belonging to the set $\{1, 2, 3, \ldots 100\}$, whose HCF with 2304 is 1, is

A. 2449

B. 1633

C. 1449

D. 2633

Sol: $2304 = 2^8 . 3^2$

Hence n cannot be multiple of 2 or 3

Then sum is $= n(1) - n(2) + n(3) - n(6)$

(Where n(a) means the sum of all numbers belonging to the set $\{1, 2, 3, \ldots 100\}$, which are divisible by a)

$= \frac{100\times101}{2} - \frac{2\times50\times51}{2} - 3 \times \frac{33\times34}{2} + 6 \times \frac{17\times16}{2}$

$= 5050 - 2550 - 1683 + 816 = 1633$

12. If f(x) is a continuous function defined as $f(x) = \begin{cases} \frac{x^3}{(1-\cos 2x)^2} \ln\left(\frac{1+\alpha x e^x}{(1+xe^x)^2}\right); x < 0 \\ \alpha; x \ge 0 \end{cases}$ then value of α is

A. $\frac{-2}{3}$

B. $\frac{2}{3}$

C. $\frac{1}{3}$

D. $\frac{-1}{3}$

Sol: $\lim_{x\to0^-} \frac{x^3}{(1-\cos 2x)^2} \ln\left(\frac{1+\alpha x e^x}{(1+xe^x)^2}\right)$

$= \lim_{x\to0^-} \frac{x^3\times x}{4\sin^4 x} \frac{\ln(1+\alpha x e^x) - 2\ln(1+xe^x)}{x}$

$= \lim_{x\to0^-} \frac{1}{4} \frac{\ln(1+\alpha x e^x) - 2\ln(1+xe^x)}{x}$

$= \lim_{x\to 0^-} \frac{1}{4}\left\{\frac{\ln(1+\alpha x e^x)\alpha e^x}{\alpha e^x} - \frac{2\ln(1+xe^x)}{\alpha e^x} e^x\right\}$

$= \lim_{x\to 0^-} \frac{1}{4}\{\alpha e^x - 2e^x\}$

$= \frac{\alpha-2}{4} = \alpha = \alpha = -\frac{2}{3}2$

13. If the domain of $f(x) = \frac{\cos^{-1}\sqrt{x^2-x+1}}{\sqrt{\sin^{-1}\left(\frac{2x-1}{2}\right)}}$ is $(\alpha + \beta]$, then the value of $\alpha + \beta =$

A. $\frac{1}{2}$

B. $\frac{3}{2}$

C. 1

D. 2

Sol: $0 \le x^2 - x + 1 \le 1$ and $0 < \frac{2x-1}{2} \le 1$

$\Rightarrow x(x-1) \le 0 \& 1 < 2x \le 3$

$\Rightarrow x \in [0,1] \cap x \in \left(\frac{1}{2}, \frac{3}{2}\right]$

$\Rightarrow x \in \left(\frac{1}{2}, 1\right]$

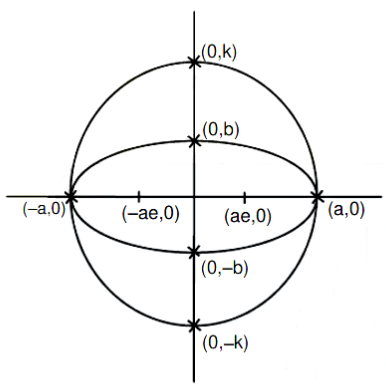
Hence $\alpha + \beta = \frac{1}{2} + 1 = \frac{3}{2}$

14. Let E_1 and E_2 be two Ellipses with same eccentricity. Where $E_1 = \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, (a > b)$ and E_2 is such that it passes through the end points of major axis of E_1 also end points of minor axis of E_1 are foci of E_2. Find the eccentricity of E_1

(1) $\sqrt{\frac{\sqrt{5}-3}{2}}$ (2) $\sqrt{\frac{\sqrt{5}+3}{2}}$ (3) $\frac{\sqrt{5}+1}{2}$ (4) $\frac{\sqrt{5}-1}{2}$

Key:- 4

Sol:-



Eccentricity of E_1 is $e \Rightarrow e^2 = 1 - \frac{b^2}{a^2}$

Eccentricity of E_2 is $e \Rightarrow e^2 = 1 - \frac{a^2}{k^2}$

So, $e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{a^2}{k^2} \Rightarrow k = \frac{a^2}{b}$ ......(i)

Also $ke = b$ .....(ii)

From equation (i) and (ii) $e = \frac{b^2}{a^2}$

Since $e^2 = 1 - \frac{b^2}{a^2} \Rightarrow e^2 = 1 - e$

$\Rightarrow e^2 + e - 1 = 0$

$\Rightarrow e = \frac{\sqrt{5} - 1}{2}$

15. Value of x for which $\left[e^x\right]^2 + \left[e^x + 1\right] - 3 = 0$ is

(1) $\left[e, e^2\right]$ (2) $\left[0, e\right]$ (3) $\left[e, \ln 2\right]$ (4) $\left[0, \ln 2\right]$

Sol:- $\left[e^x\right]^2 + \left[e^x + 1\right] - 3 = 0$

$\left[e^x\right]^2 + \left[e^x\right] - 2 = 0$

Let $\left[e^x\right] = t$

$t^2 + t - 2 = 0$

$(t+2)(t-1) = 0$

$t = 1, -2$

$\left[e^x\right] : 1, -2$ (-2 is not possible)

$\left[e^x\right] = 1$

$x \in \left[0, \ln 2\right]$

16. Evaluates $\int \frac{e^x\left(2 - x^2\right)}{(1-x)\sqrt{1-x^2}} dx$

(1) $e^x\sqrt{\frac{1+x}{1-2x}}+C$ (2) $e^x\sqrt{\frac{1+x}{1-x}}+C$ (3) $e^x\sqrt{\frac{1-x}{1+x}}+C$ (4) $e^x\left(\frac{1+x}{1-x}\right)+C$

Key:- 2

Sol:- $I=\int\frac{e^x\left(2-x^2\right)}{(1-x)\sqrt{1-x^2}}dx$

$$=\int e^x\left\{\frac{1}{(1-x)^{3/2}(1+x)^{1/2}}+\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}\right\}dx$$

$$f(x)=\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}\Rightarrow f'(x)=\frac{1}{(1-x)^{3/2}(1+x)^{1/2}}$$

$$\Rightarrow I=\int e^x\left(f'(x)+f(x)\right)dx$$

$$=e^x f(x)+C=e^x\sqrt{\frac{1+x}{1-x}}+C$$

17. Four dice are rolled and the outcomes are put in 2×2 matrices. Find the probability that such a matrix will be non singular and all its entries are different

(1) $\frac{71}{81}$ (2) $\frac{80}{81}$ (3) $\frac{30}{71}$ (4) $\frac{25}{71}$

Key:- 2

Sol:- $X=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$|x|=ad-bc=0$

$$\left.\begin{matrix}(1,6) & (3,2)\\ (3,4) & (6,2)\end{matrix}\right\}8+8 \text{ possibilities}$$

$$\text{Required probability}=1-\frac{16}{6^4}=\frac{80}{81}$$

18. Find the values of λ & μ for which the system of equations

$x+y+z=6$

$3x+5y+5z=26$

$x+2y+\lambda z=\mu$

(1) $\lambda = 2, \mu \neq 10$ (2) $\lambda \neq 2, \mu = 10$ (3) $\lambda \neq 3, \mu = 10$ (4) $\lambda \neq 2, \mu \neq 10$

Key:- 1

Sol:-

19. The number of elements in the set $\{x \in R : (|x| - 3)|x - 4| = 6\}$ is equal to

(1) 3 (2) 4 (3) 2 (4) 1

Key:- 3

Sol: $(|x| - 3)\,|x - 4| = 6$

Case 1: $x \geq 4$

$(x-3)(x-4) = 6$

$x^2 - 7x + 6 = 0$

$(x-1)(x-6) = 0$

$x = 1, 6 \Rightarrow x = 6$

Case 2: $0 < x < 4$

$(x-3)(4-x) = 6$

$x^2 - 7x + 18 = 0$

$D < 0$ No solution

Case 3: $x \leq 0$

$(-x-3)(4-x) = 6$

$(x+3)(x-4) = 6$

$x^2 - x + 18 = 0$

$x = \frac{1 \pm \sqrt{73}}{2}$

$x = \frac{1 - \sqrt{73}}{2}$

20. If $\int_0^{100\pi} \frac{\sin^2 x}{e^{\left(\frac{x}{\pi} - \left[\frac{x}{\pi}\right]\right)}} dx = \frac{\alpha \pi^3}{1 + 4\pi^2}, \alpha \in R$ where $[x]$ is the greatest integer function, then α is

(1) $50(e-1)$ (2) $150(e^{-1}-1)$ (3) $200(1-e^{-1})$ (4) $100(1-e)$

Key:- 3

Sol:- $\int_0^{100} \frac{\sin^2 x}{e^{\left\{\frac{x}{\pi}\right\}}} dx$

$\Rightarrow 100 \int_0^{\pi} \frac{\sin^2 x}{e^{\left\{\frac{x}{\pi}\right\}}} dx$

$\Rightarrow 50 \int_0^{\pi} e^{\left\{\frac{x}{\pi}\right\}} [1 - \cos 2x] \ dx$

$\Rightarrow 50 \left[e^{-x/\pi} \times (-\pi)\right]_0^{\pi} - 50 \int_0^{\pi} e^{-x/\pi} \cos 2x \, dx$

$\Rightarrow 50 \times (-\pi)(e^{-1} - 1) - \frac{50 \times \left[e^{-x/\pi}\left(\frac{-1}{\pi} \times \cos 2x + 2\sin 2x\right)\right]_0^{\pi}}{\left(\frac{1}{\pi^2} + 4\right)}$

$\Rightarrow -50\pi(e^{-1} - 1) - \frac{50\pi^2}{(1+4\pi^2)}\left[e^{-1}\left(\frac{-1}{\pi}\right) + \frac{1}{\pi}\right]$

$\Rightarrow \frac{200\,\pi^3\,(1 - e^{-1})}{1+4\pi^2}$

So, $\alpha = 200(1 - e^{-1})$