Mathematics

MCQ TYPE

1. Let E : $y^2 = 8x$. Two tangents PQ and PQ' are drawn from point P(–2, 4) to curve then which of the following is/are true (F be focus of parabola)

(A) $\Delta PQQ'$ is right angle Δ

(B) ΔPFQ is right angle Δ

(C) Line joining QQ' passes through F

(D) $PF = 5\sqrt{2}$

Ans. (B,C)

Sol. Point P(–2, 4) lies on directrix

$PF = 4\sqrt{2}$

By property option B,C are true

2. f(1) = 1

$$\int_0^{\frac{\pi}{3}} f(x)dx = 0$$

(A) $f(x) - 3\sin 3x = \frac{-6}{\pi}$ has solution in $\left[0, \frac{\pi}{3}\right]$

(B) $f(x) - 3\cos 3x = 0$ has solution in $\left[0, \frac{\pi}{3}\right]$

(C) $\lim_{x \to 0} \frac{x\int_0^x f(t)dt}{1 - e^{x^2}} = -1$

(D) $\lim_{x \to 0} \frac{x\int_0^x f(t)dt}{1 - \cos(3x)} = -1$

Ans. **(B,C)**

Sol. Option (A)

$$g(x) = \int_0^x \left(f(x) - 3\sin 3x + \frac{6}{\pi}\right)dx$$

$$g(0) = 0,\ g\left(\frac{\pi}{3}\right) = -2 + 2 = 0$$

Option(B)

$$g(x) = \int_0^x (f(x) - 3\cos 3x)dx$$

$g(0)=0,\ g\left(\frac{\pi}{3}\right)=0$

Option(C)

$$\lim_{x\to 0}\frac{x\int_0^x f(t)dt}{\left(\frac{\left(1-e^{x^2}\right)}{x^2}\right)x^2}=\lim_{x\to 0}-\frac{\int_0^x f(t)dt}{x}=-f(0)=-1$$

Option (D)

$$\lim_{x\to 0}\frac{x\int_0^x f(t)dt}{\left(\frac{(1-\cos 3x)}{9x^2}\right)9x^2}=\lim_{x\to 0}\frac{2\int_0^x f(t)dt}{9x}=\frac{+2}{9}$$

3. Given $\overrightarrow{OA}=2\hat{i}+2\hat{j}+\hat{k}$

$\overrightarrow{OB}=\hat{i}-2\hat{j}+2\hat{k}$

and $\overrightarrow{OC}=\frac{1}{2}\left(\overrightarrow{OB}-\lambda\overrightarrow{OA}\right)$

if $\left|\overrightarrow{OB}\times\overrightarrow{OC}\right|=\frac{9}{2}$ Where A,B,C are non collinear points then

(A) area of $\Delta ABC=\frac{9}{2}$

(B) Projection of $\overrightarrow{OC}$ on $\overrightarrow{OA}$ is $\frac{3}{2}$

(C) area of $\Delta OAB=\frac{9}{2}$

(D) Projection of $\overrightarrow{OC}$ on $\overrightarrow{OA}$ is $\frac{1}{2}$

Ans. **(A,B,C)**

Sol. $\left|\overrightarrow{OB}\times\overrightarrow{OC}\right|=\frac{9}{2}$

$\Rightarrow \left|\overrightarrow{OB}\times\left(\overrightarrow{OB}-\lambda\overrightarrow{OA}\right)\right|=9$

$\Rightarrow \left|\lambda\left(\overrightarrow{OB}\times\overrightarrow{OA}\right)\right|=9$

$\Rightarrow |\lambda|=1 \qquad \Rightarrow \lambda\pm 1$

For $\lambda=-1$, point A,B,C are collinear. Hence, $\lambda=1$

$\Rightarrow \overrightarrow{OC}=\frac{1}{2}\left(\overrightarrow{OB}-\overrightarrow{OA}\right)$

$\Rightarrow \overrightarrow{OC} = \frac{1}{2}\left(-\hat{i} - 4\hat{j} + \hat{k}\right)$

$\Rightarrow \overrightarrow{AB} = -\hat{i} - 4\hat{j} + \hat{k}$

and $\overrightarrow{AC} = \frac{-5}{2}\hat{i} - 4\hat{j} - \frac{1}{2}\hat{k}$

area of $(\Delta OAB) = \frac{1}{2}\left|\overrightarrow{OA} \times \overrightarrow{OB}\right| = \frac{9}{2}$

area of $(\Delta ABC) = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{9}{2}$

Projection of OC on OA = $\left|\frac{\overrightarrow{OC}.\overrightarrow{OA}}{\overrightarrow{OA}}\right| = \frac{3}{2}$

4. S_1 : i, j, k such that i, j, k $\in$ {1,2,-----10}

S_2 : i, j, k such that $1 \le i < j+2 \le 10$. & $i, j \in \{1, 2, ----10\}$

S_3 : i, j, k, ℓ such that $1 \le i < j < k < \ell \le 10$. & $i, j, k, \ell \in \{1, 2, ----10\}$

S_4 : i, j, k, ℓ such that all are distinct and $i, j, k, \ell \in \{1, 2, ----10\}$

The number of element in S_i is n_i

(A) $n_3 = 220$ (B) $\frac{n_4}{12} = 420$ (C) $n_2 = 28$ (D) $n_1 = 10^3$

Ans. **(B,C,D)**

Sol. $S_1 : 10^3$

S_2 : i j

1 7 choice

2 6 choice

⋮

7 ,

$S_2 = 1 + 2 + - ---. 7 = \frac{7(7+1)}{2} = 28$

$S_3 = {}^{10}C_4 = 210$

$S_4 = {}^{10}C_4 \times 4! = 420$

5. Let $\frac{dy}{dx}+xy=xe^{\beta x}$ passes through (1,1), then

(A) $ye^{-x}=\frac{x^2-1}{2}+\frac{1}{e}$ is a solution to this equation.

(B) $ye^{-x}=\frac{x^2+1}{2}+\frac{1}{e}-1$ is a solution to this equation.

(C) $ye=\frac{e^{2x}}{4}(2x-1)+e-\frac{e^2}{4}$ is a solution to this equation.

(D) $ye^{x}=\frac{e^{2x}}{4}(2x+1)+e$ is a solution to this equation.

Ans. **(A, C)**

Sol. $\frac{dy}{dx}+\alpha y=xe^{\beta x}$

I.F = $e^{\alpha x}$

So $ye^{\alpha x}=\int xe^{(\alpha+\beta)x}dx$

If $\alpha+\beta=0$ $\qquad ye^{\alpha x}=\frac{x^2}{2}+C$

passing through (1,1), So $ye^{\alpha x}=\frac{x^2}{2}-\frac{1}{2}+e^{\alpha}$.

If $\alpha+\beta\neq 0$ $\qquad ye^{\alpha x}=\frac{xe^{(\alpha+\beta)x}}{\alpha+\beta}-\int\frac{e^{(\alpha+\beta)x}dx}{\alpha+\beta}$

$\Rightarrow ye^{\alpha x}=\frac{xe^{(\alpha+\beta)x}}{\alpha+\beta}-\frac{e^{(\alpha+\beta)x}dx}{(\alpha+\beta)^2}+C$

$\Rightarrow ye^{\alpha x}=\frac{e^{(\alpha+\beta)x}}{(\alpha+\beta)^2}\{(\alpha+\beta)x-1\}+e^{\alpha}-\frac{e^{\alpha+\beta}}{(\alpha+\beta)^2}(\alpha+\beta-1)$

NUMERICAL

6. A number is chosen random from set {1,2,3,..........,2000}. Let p be the probability that chosen number is multiple of 3 or 7. then 500 p is equal to _______.

Ans. **(214)**

Sol. Number multiple of 3 are $\left[\frac{2000}{3}\right] = 666$

Number multiple of 7 are $\left[\frac{2000}{7}\right] = 285$

Number multiple of 21 are $\left[\frac{2000}{21}\right] = 95$

$$p = \frac{666+285-95}{2000} = \frac{856}{2000}$$

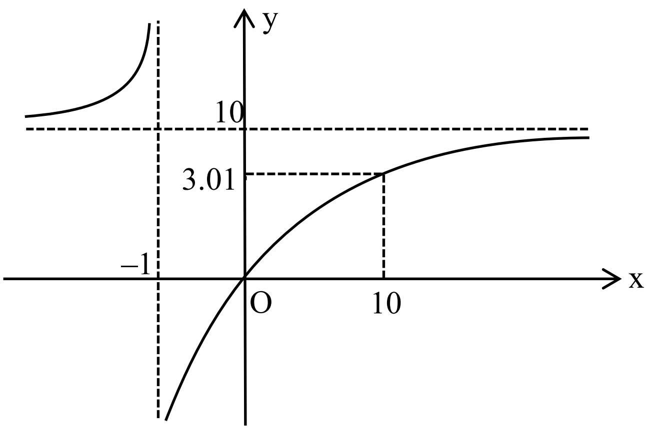
$500\ p = 214$

7. Let $I = \int_0^{10}\left[\sqrt{\frac{10x}{x+1}}\right]dx$, then find the value of 9I.

Ans. **(182)**

Sol. $y = \frac{10x}{x+1} = 10 - \frac{10}{(x+1)}$

$$\frac{dy}{dx} = \frac{10}{(x+1)^2}$$



$$\frac{10x}{x+1} = 1 \Rightarrow x = \frac{1}{9}$$

$$\frac{10x}{x+1} = 4 \Rightarrow x = \frac{2}{3}$$

$$\frac{10x}{x+1} = 9 \Rightarrow x = 9$$

$$\int_0^{10}\left[\sqrt{\frac{10x}{x+1}}\right]dx = \int_0^{1/9} 0\ dx + \int_{1/9}^{2/3} dx + \int_{2/3}^{9} 2\ dx + \int_9^{10} 3\ dx$$

$$= \left(\frac{2}{3}-\frac{1}{9}\right)+2\left(9-\frac{2}{3}\right)+3$$

$$= \frac{5}{9}+\frac{50}{3}+3$$

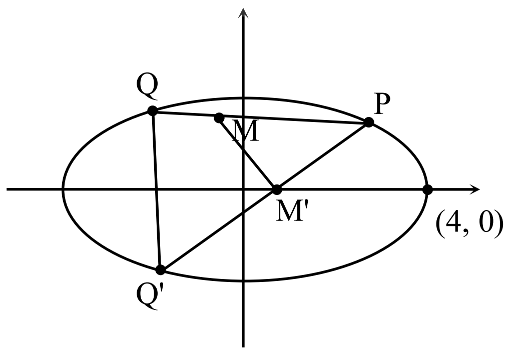
$$= \frac{182}{9}$$

$\therefore\ 9I = 182$

8. Let E be the ellipse $\frac{x^2}{16}+\frac{y^2}{9} = 1$. For any 3 distinct points P, Q, Q' on E, Let M(P, Q) be mid point of line segment joining P and Q and M'(P, Q') be mid point of line segment joining P and Q'. then maximum possible value of the distance between M(P, Q) and M'(P, Q') as P, Q and Q' vary on E is ________

Ans. (4)

Sol.



$$MM' = \frac{1}{2}QQ'$$

Maximum distance between QQ' is 8

Hence, maximum distance between M and M' is 4

Paragraph for Question Nos. 9 to 10

$f_1 : (0, \infty) \to IR$, $f_2 : (0, \infty) \to IR$ is defined by $f_1(x) = \int_0^x \prod_{i=1}^{21} (t-i)^i \, dt$, $x > 0$,

$f_2 = 98(x-1)^{50} - 600(x-1)^{49} + 2450$, $x > 0$ where , for any positive number n in R, numbers $a_1, a_2, \ldots\ldots, a_n$, $\prod_{i=1}^{n} a_i$ denotes the product of $a_1, a_2, \ldots\ldots, a_n$. Let m_i and n_i respectively denote number of points of local minima and number of points of local maxima of function f_i, $i = 1, 2$ in the interval $(0, \infty)$

9. Value of $2m_1 + 3n_1 + m_1n_1$ is

Ans. **(57)**

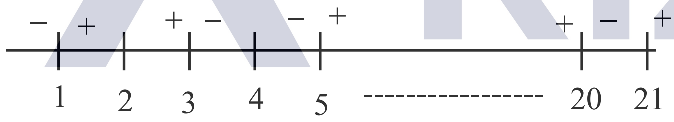
10. Value of $6m_2 + 4n_2 + 8m_2n_2$ is

Ans. **(6)**

Sol. **(2 to 3)**

(a) $f_1(x) = \int_0^x (t-1)(t-2)^2 ------(t-21)^{21} dt$

$\Rightarrow f_1'(x) = (x-1)(x-2)^2 ---------(x-21)^{21}$



At all odd integers from 1 to 21 f(x) will have an extrema with 1,5,9, 13, 17,21 being points of minima & 3, 7, 11, 15, 19 being points of maxima

So $m_1 = 6$ & $n_1 = 5$

Hence $2m_1 + 3n_1 + m_1n_1 = 57$

(b) $f_2'(x) = 98 \times 50 (x-1)^{49} - 600 \times 49 (x-1)^{48}$

$= 4900 (x-1)^{48} ((x-1) - 6$

$= 4900 (x-1)^{48} (x-7)$

So extrema is at x = 7 only . which is minima

$m_2 = 1$, $n_2 = 0$

Hence $6m_2 + 4n_2 + 8m_2n_2 = 6$

Paragraph for Question Nos. 11 to 12

Let $g_i : \left[\frac{\pi}{8}, \frac{3\pi}{8}\right] \to R \quad \forall\ i = 1, 2$ and $f : \left[\frac{\pi}{8}, \frac{3\pi}{8}\right] \to R$ be function such that $g_1(x) = 1$, $g_2(x) = |4x - \pi|$ and $f(x) = \sin^2 x \quad \forall\ x \in \left[\frac{\pi}{8}, \frac{3\pi}{8}\right]$.

If $S_i = \int_{\pi/8}^{3\pi/8} f(x) g_i(x) dx$, for i = 1, 2 then

11. The value of $\frac{48S_2}{\pi^2}$ is

Ans. **(1.5)**

12. The value of $\frac{16S_1}{\pi}$ is

Ans. **(2)**

Sol. **(4 to 5)**

$$S_1 = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} f(x) g_1(x)$$

$$\Rightarrow S_1 = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \sin^2 x dx \qquad (1)$$

$$\Rightarrow S_1 = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \sin^2\left(\frac{\pi}{8} + \frac{3\pi}{8} - x\right) dx$$

$$= \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \cos^2 x\, dx \qquad (2)$$

add (1) & (2) we get

$$2S_1 = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} 1\, dx = \frac{2\pi}{8} \Rightarrow S_1 = \frac{\pi}{8}$$

$$\Rightarrow \frac{16S_1}{\pi} = 2$$

Now $S_2 = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \sin^2 |4x - \pi| dx$ (3)

$\Rightarrow S_2 = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \cos^2 \left|4\left(\frac{\pi}{2} - x\right)\right| dx$

$\Rightarrow S_2 = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \cos^2 |4x - \pi| dx$ (4)

add (3) & (4) we get

$2S_2 = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} |4x - \pi| dx = 2 \times \frac{1}{2} \times \frac{\pi}{8} \times \frac{\pi}{2} = \frac{\pi^2}{16}$

$\frac{48S_2}{\pi^2} = \frac{3}{2} = 1.5$

Paragraph for Question Nos. 13 to 14

Consider region R = {(x, y) ∈ R × R : $x \geq 0$ and $y^2 \leq 4 - x$}. Let F be the family of all circles that are connected in R and have centers on the x-axis. Let C be the circle that has largest radius among the circles in F. Let (α, β) be a point where the circle C meets the curve $y^2 = 4 - x$ then

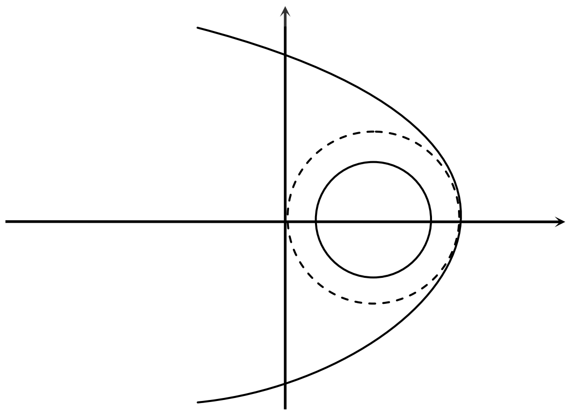
13. Radius of circle C

Ans. **(4)**

14. α is equal to

Ans. **(0)**

Sol. **(6 to 7)**



Note that largest circle will be touching the parabola and all point on the circle should have x-ordinates ≥ 0

Now normal to this curve at $P(4 - t^2, -2t)$ meets x axis at $Q(2 - t^2, 0)$

If $t^2 > 0$, then the circle cannot touch the parabola, else some part of it will have points whose x-coordinates are less than O

So for maximum radius t = 0, radius = 2 and the circle touches at (0, 0)

Paragraph for Question Nos. 15 to 16

$x^2 + y^2 = r^2$

$a_k = \frac{1}{2^{k-1}}$

$S_n = \sum_{k=1}^{n} a_k$, $S_0 = 1$

Let C_n be a circle whose contre is $(S_{n-1}, 0)$ and radius is a_n. Let D_n be a circle whose contre is (S_{n-1}, S_{n-1}) and radius is a_n.

15. If $r = \left(\frac{2^{199}-1}{2^{198}}\right)\sqrt{2}$, then number of circles D_n which completely be inside this circle

Ans. **(199)**

Sol. $\sqrt{2}\, S_{n-1} + a_n < \left(\frac{2^{199}-1}{2^{198}}\right)\sqrt{2}$

$$\sqrt{2}\left(2-\frac{1}{2^{n-2}}\right)+\frac{1}{2^{n-1}} < \left(\frac{2^{199}-1}{2^{198}}\right)\sqrt{2}$$

$$2\sqrt{2}-\frac{1}{2^{n-2}}\sqrt{2}+\frac{1}{2^{n-1}} < 2\sqrt{2}-\frac{\sqrt{2}}{2^{198}}$$

$$\frac{1}{2^{n-2}}\left(\frac{1}{2}-\sqrt{2}\right) < -\frac{\sqrt{2}}{2^{198}}$$

$$\frac{1}{2^{n-2}}\frac{(2\sqrt{2}-1)}{2} > \frac{\sqrt{2}}{2^{198}}$$

$$2^{n-2} < \left(2-\frac{1}{\sqrt{2}}\right)2^{197}$$

$n - 2 \le 197$

$n \le 199$

number of circle = 199

16. If $r = \frac{1025}{513}$, find number of circles C_n which completely be inside this circle

Ans. **(10)**

Sol. $S_{n-1} + a_n < \frac{1025}{513}$

$\Rightarrow a_1 + a_2 + ... + a_n < \frac{1025}{513}$

$\Rightarrow 1 - \left(\frac{1}{2}\right)^n < \frac{1025}{1026}$

$\Rightarrow 2n < 1026 \Rightarrow n \le 10$

hence number of circles = 10

Paragraph for Question Nos. 17 to 18

If $\Psi_1 : [0, \infty) \to R$

$\Psi_2 : [0, \infty) \to R$

$f : [0, \infty) \to R$

$g : [0, \infty) \to R$

If $f(0) = g(0) = 0$ & $\Psi_1(x) = e^{-x} + x$

$\Psi_2(x) = x^2 - 2x - 2e^{-x} + 2$

$$f(x) = \int_{-x}^{x} (|t| - t^2)e^{-t^2}dt, \ x > 0$$

$$g(x) = \int_{0}^{x^2} \sqrt{t}\, e^{-t}\, dt, \text{ then}$$

17. There exist a β

(A) $\beta \in (0, x)$ such that $\Psi_2(x) = 2x(\Psi_1(\beta) - 1)$

(B) For every $x > 1$, there exist a α, a $\in (1, x)$ such that $\Psi_1(x) = 1 + \alpha x$

(C) $f\left(\sqrt{\ln 3}\right) + g\left(\sqrt{\ln 3}\right) = \frac{1}{3}$

(D) None of these

Ans. **(A)**

Sol. (A) $\Psi'_2(x) = 2\Psi_1(x) - 2$

from LMVT, $\frac{\Psi_2(x) - \Psi_2(0)}{x - 0} = \Psi'_2(\beta)$ for atleast one $\beta \in (0, x)$

$\Rightarrow \Psi_2(x) = 2x(\Psi_1(\beta) - 1)$

(B) for $\alpha \in (1, x)$, $\Psi_1(x) - 1 - \alpha x = 0$

$\Rightarrow e^{-x} + x - 1 - \alpha x = 0$

$\Rightarrow (e^{-x} - 1) = x(\alpha - 1)$

Which is not possible because LHS < 0 & RHS > 0

(C) $f(x) = 2\int_{0}^{x}(t - t^2)e^{-t^2}dt$; $x > 0$

$g(x) = \int_{0}^{x^2}\sqrt{t}\, e^{-t}dt$; $x > 0$

put $t = u^2$

$\therefore\ g(x) = 2\int_{0}^{x} u^2 e^{-u^2}du = 2\int_{0}^{x} t^2 e^{-t^2}dt$

now, $f(x) + g(x) = \int_{0}^{x} 2te^{-t^2}dt = 1 - e^{-x^2}$

$\Rightarrow f(\sqrt{\ln 3}) + g(\sqrt{\ln 3}) = 1 - e^{-\ln 3} = \frac{2}{3}$

18. Which of the following is correct

(A) $\phi_1(x) < 1 \ \forall \ x \in (0, \infty)$

(B) $\phi_2 (x) < 0 \ \forall \ x \in (0, \infty)$

(C) $f(x) > 1 - e^{-x^2} - \left(\frac{2}{3}x^3 - \frac{2}{5}x^5\right), x \in \left(0, \frac{1}{2}\right)$

(D) $g(x) \le \frac{2}{3}x^3 - \frac{2}{5}x^5 + \frac{1}{7}x^7, \ x \in \left(0, \frac{1}{2}\right)$

Ans. **(D)**

Sol. (A) $e^{-x} + x < 1$ for $x \in (0, \infty)$ is incorrect

LHS is increasing and unbounded function

(B) $x^2 - 2x - 2e^{-x} + 2 < 1$

for $x \in (0, \infty)$ is incorrect because LHS $\to \infty$ when $x \to \infty$

(C) Now $f(x) + g(x) = 1 - e^{-x^2}$

$\Rightarrow f(x) = 1 - e^{-x^2} - g(x)$

$\Rightarrow f(x) \le 1 - e^{-x^2} - \left(\frac{2}{3}x^3 - \frac{2}{3}x^5\right)$

(D) $g(x) = \int_0^{x^2} \sqrt{t}e^{-t}dt$

$\Rightarrow g(x) \le \int_0^{x^2} \sqrt{t}\left(1 - t + \frac{t^2}{2}\right) dt \Rightarrow g(x) \le \frac{2}{3}x^3 - \frac{2}{5}x^5 + \frac{1}{7}x^7$

$g(x) \ge \int_0^{x^2} \sqrt{t}(1-t)\, dt \Rightarrow g(x) \ge \frac{2}{3}x^3 - \frac{2}{5}x^5$